SMOOTHNESS AND DISCONTINUITIES OF WEAK SOLUTIONS TO PARABOLIC SYSTEMS (Variational Problems and Related Topics)

Stara, J.; John, O.

数理解析研究所講究録
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Smoothness and discontinuities of weak solutions to parabolic systems

J. Stará, O. John

The aim of this note is to give a survey of several recent results dealing with regularity of weak solutions to parabolic systems (existence of L_2–time derivative, Hölder continuity in two dimensional space) and to illustrate that in three dimensions parabolic systems do not conserve, in general, the regularizing property of the heat equation.

In the first part, we give a comparison of results about elliptic and parabolic systems and list some open problems. Second part is devoted to examples of non-regular solutions and the third part contains partial positive results valid under additional assumptions on coefficients.

1. Comparison of results for elliptic and parabolic systems

We are interested in quasilinear elliptic systems of the form

$$D_\alpha(A_{ij}^{\alpha\beta}(x,u)D_\beta u^j)=0, i=1,...,M \text{ on } \Omega; \tag{1.1}$$

and their parabolic counterpart

$$u_t^i=D_\alpha(A_{ij}^{\alpha\beta}(x,t,u)D_\beta u^j), i=1,...,M \text{ on } Q. \tag{1.2}$$

(The summation convention is used throughout the paper.) Domain Ω is considered to be a nonempty open subset of $\mathbb{R}^m, Q=\Omega\times(0,T)$ for a positive T. $A_{ij}^{\alpha\beta}(i,j=1,...,M,\alpha,\beta=1,...,m)$ are uniformly bounded Carathéodory functions satisfying in case (1.1) ellipticity condition and its analogy[1]

$$\begin{aligned}&\exists\ \lambda_0,\lambda_1\in(0,\infty)\ \forall\ \xi\in\mathbb{R}^{m.M}\ \forall\ u\in\mathbb{R}^M\\ &\text{for almost every } x\in\mathbb{R}^m \text{ for almost every } t\in(0,T)\\ &\lambda_0|\xi|^2\le\langle A(x,t,u)\xi,\xi\rangle\le\lambda_1|\xi|^2.\end{aligned} \tag{1.3}$$

in the parabolic case (1.2). For simplicity case we shall deal mainly with interior regularity.

Elliptic systems, linear case

If $A_{ij}^{\alpha\beta}$ depend only on x and are continuous on their domain, then according to classical results of C. B.Morrey [6], A. Douglis, L. Nirenberg [2] every weak solution of (1.1) is locally Hölder continuous. The proof of Theorem 3.1 in [3] indicates that the continuity of coefficients in one point implies the Hölder continuity of any weak solution in a neighbourhood of this point.

[1] We denoted by $\langle u,v\rangle$ scalar product in any finite dimensional space $\mathbb{R}^p, p\in\mathbb{N}, |u|=\langle u,u\rangle^{\frac{1}{2}}$.

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On the other hand, for $m \geq 3$ the discontinuity of coefficients in one point can cause the discontinuity (even unboundedness) of a solution (see [1]). The counterexample of J. Souček (see[7]) gives a solution of (1.1) which is discontinuous on a dense countable subset. Moreover, for every set $F \subset \mathbb{R}^m$ of the type F_σ there is a system (1.1) and its solution u which is bounded, essentially discontinuous at all points of F and essentially continuous at all points of $\mathbb{R}^m - F$. (See [8].)

In any dimension we have a $W^{1,p}$ estimate of solutions for some (sufficiently small) $p > 2$ (see e.g. [5], [3]). If $m = 2$, this estimate and embedding theorems guarantee Hölder continuity of solutions.

Parabolic systems, linear case

In this case, too, any weak solution (i.e., any locally square integrable function with locally square integrable space gradient satisfying the system in the sense of distributions) is Hölder continuous on a neighbourhood of any point of continuity of coefficients (see [12], [19]).

Any weak solution of an elliptic system (1.1) can be considered as a stationary solution to a parabolic system (1.2). Thus elliptic examples can be interpreted as stationary parabolic problems on Q. It would indicate singularities of solutions appearing on cylindrical subsets of $\mathbb{R}^{m+1}$. It is more interesting to ask whether a weak solution of a parabolic system can develop a singularity in the interior of space-time cylider starting from smooth initial data. If $m \geq 3$ this situation can occur (see [9] and part 2 of this paper) eventhough much less is known about possible structure of a singular set (see [10]).

For parabolic systems L_p estimates of space gradient for sufficiently small $p > 2$ (see [11], [13], [15], part 3 of this paper) hold, too. However even for $m = 2$ they do not imply Hölder continuity of solutions. As far as we know the question whether for $m = 2$ any weak solution to a linear parabolic system with L_∞ coefficients is locally Hölder continuous is open.

2. Examples

Theorem 2.1 (see [9]) *Let $n \geq 3, \kappa \in (0, 2(n-1)(n-2))$; for $x \in \mathbb{R}_n, t \in (-\infty, 1)$ put*

$$u(x,t) = \frac{x}{\sqrt{\kappa(1-t) + |x|^2}}.$$

Then u is real analytic on $\mathbb{R}_n \times (-\infty, 1)$ and solves a quasilinear parabolic system

$$u_t^i = D_\alpha(A_{ij}^{\alpha\beta}(u) D_\beta u^j, i = 1, ..., n \tag{2.1}$$

with real analytic coefficients $A_{ij}^{\alpha\beta}(u)$ on a neighbourhood of $\overline{B(0,1)}$.

The coefficients are given by the formula

$$A_{ij}^{\alpha\beta}(u) = \theta\delta_{ij}\delta\alpha\beta + A_{i\alpha}(u)A_{j\beta}(u)$$

with

$$(2.2)\qquad A_{i\alpha}(u)=\frac{\{n-1-\theta-|u|^2(1+\frac{\kappa}{2(n-1)})\}\delta_{i\alpha}+(1+\theta+\frac{\kappa}{2(n-1)})u^iu^\alpha}{\sqrt{n(n-1-\theta)-\{2(n-1-\theta)+\frac{\kappa}{2}\}|u|^2-\theta|u|^4}}.$$

For $\theta\in(0,n-2-\frac{\kappa}{2(n-1)})$ the expression under the squareroot in the denominator is positive on $\overline{B(0,1)}$. The coefficients are then real analytic on the same set and satisfy ellipticity condition

$$\lambda_0|\xi|^2\leq\langle A(u)\xi,\xi\rangle\leq\lambda_1|\xi|^2$$

where

$$\lambda_0=\theta,\ \lambda_1=\frac{(n-1)^2-\theta(n-2-\frac{\kappa}{2(n-1)})}{n-2-\frac{\kappa}{2(n-1)}-\theta}.$$

Denote $r>1$ radius of a ball on which the denominator in (2.2) is positive and let $\Phi\in C^\infty(\mathbb{R})$ be any function such that $0\leq\Phi\leq1$ on $\mathbb{R},\Phi(s)=0$ for $s\geq r,\Phi(s)=1$ for $|s|<\frac{1+r}{2}$. Put

$$\tilde{A}_{ij}^{\alpha\beta}(u)=\begin{cases}\theta\delta_{ij}\delta\alpha\beta+\Phi(|u|^2)A_{i\alpha}(u)A_{j\beta}(u) & \text{for}|u|<r\\ \theta\delta_{ij}\delta\alpha\beta & \text{otherwise.}\end{cases}$$

Then $\tilde{A}_{ij}^{\alpha\beta}$ are infinitely differentiable on $\mathbb{R}^n$, system with these coefficients satisfies ellipticity condition with the same λ_0,λ_1 and admits the same solution u.

Inserting values of u in $\tilde{A}_{ij}^{\alpha\beta}(u)$ we see that u solves also a linear parabolic system with coefficients which are bounded, real analytic on $(\mathbb{R}^n\setminus\{0\})\times(-\infty,1)$ and can be extended by different ways on $\mathbb{R}^{n+1}$ as bounded and measurable functions. Thus the discontinuity of a solution can disappear for $t>1$ or can survive for any time interval.

By analogous procedure as in the first quasilinear case we obtain examples of L_∞ blow up for linear parabolic system.

Theorem 2.2 (see[9])*Let* $n\geq3,\gamma\in(0,min(\sqrt{n-1}-1,\frac{1}{2}),\kappa\in(0,2(n-1)(n-2-2\gamma))$; *for* $x\in\mathbb{R}_n,t\in(-\infty,1)$ *put*

$$u(x,t)=\frac{x}{|x|^\gamma\sqrt{\kappa(1-t)+|x|^2}}.$$

Then u *is Hölder continuous on* $\mathbb{R}_n\times(-\infty,1)$ *and it is a weak solution of a linear parabolic system*

$$u_t^i=D_\alpha(A_{ij}^{\alpha\beta}(x,t)D_\beta u^j,i=1,...,n$$

with $A_{ij}^{\alpha\beta}\in L_\infty(\mathbb{R}^n\times(-\infty,1)$ *satisfying uniform ellipticity condition*

$$\exists\ \lambda_0,\lambda_1\in(0,\infty)\ \forall\ \xi\in\mathbb{R}^{n^2}\ \forall\ x\in\mathbb{R}^n\ \forall\ t\in(-\infty,1)$$
$$\lambda_0|\xi|^2\leq\langle A(x,t)\xi,\xi\rangle\leq\lambda_1|\xi|^2.$$

Nevertheless,

$$\lim_{t \to 1-} ||u(.,t)||_{L_\infty(\mathbb{R}^n)} = \infty.$$

The question of how "large" the sets of singular points of a solution to nonsmooth parabolic system can be is not completely solved. Partial answer to this question is given in [10].

3. Regularity

In this part we concentrate mainly on two points, i.e. the existence of time derivative and Hölder continuity of solutions.

Nečas and Šverák in [13] considered a nonlinear system

$$u_t^i = D_\alpha(a_i^\alpha(\nabla u), i = 1, ..., M \text{ on } Q. \tag{3.1}$$

with continuously differentiable coefficients a_i^α and proved that $u \in C_{loc}^{1,\mu}(Q)$ if $m = 2$ and $u \in C_{loc}^{0,\mu}(Q)$ if $m \leq 4$.

In 1995, Gröger and Rehberg (see [16]) considered the system

$$u_t^i - D_\alpha(A_{ij}^{\alpha\beta}(x,t,u)D_\beta u^j) = f^i, i = 1, ..., M \text{ on } Q. \tag{3.2}$$

They solved initial and boundary value problem for this system for sufficiently small time T in a space, which for $m = 2$ is embedded in $C^{0,\mu}(Q)$. Coefficients $A_{ij}^{\alpha\beta}$ are supposed to be uniformly continuous in t and bounded and measurable in space variables. Under these assumptions the time derivative belongs to $L_q((0,T); W^{-1,p})$ for a $p > 2$.

In 1997 Naumann, Wolff and Wolf in [17] proved that if we suppose coefficients in (3.2) to be μ-Hölder continuous with $\mu > \frac{1}{2}$ (sufficiently near to 1) and $m = 2$ then $u \in C^{0,\mu}(Q)$ and there is a $p > 2$ such that $u_t \in L_p((0,T); L_2(\Omega))$.

In 1996 we proved in [15] that if coefficients a_i^α are Lipschitz continuous in t and bounded and measurable in space variables and $m = 2$, then all solutions of

$$u_t^i - D_\alpha(a_i^\alpha(x,t,u,\nabla u) = f^i, i = 1, ..., M \text{ on } Q. \tag{3.3}$$

are Hölder continuous and there is a $p > 2$ such that $u_t \in L_\infty((0,T); L_p(\Omega))$.

If we drop the assumption $m = 2$ we obtain a result which is a slight generalization of [18]:

Theorem 3.1(see [14]) *Let $f^i \in L_2(Q)$, coefficients $a_i^\alpha(x,t,u,p)$ be Carathéodory functions continuously differentiable in u, p and satisfy on their domains*

(i) growth conditions:

$$|a_i^\alpha(x,t,u,p)| \leq M(1 + |u| + |p|),$$
$$|\frac{\partial a_i^\alpha}{\partial u^j}(x,t,u,p)| + |\frac{\partial a_i^\alpha}{\partial p_\beta^j}(x,t,u,p)| \leq M,$$

(ii) ellipticity condition:

$$\frac{\partial a_i^\alpha}{\partial p_\beta^j}(x,t,u,p)\xi_\alpha^i\xi_\beta^j \geq \lambda_0|\xi|^2,$$

(iii) Hölder continuity in t: for $\gamma \in (\frac{1}{2}, 1]$

$$|a_i^\alpha(x,t_1,u,p) - a_i^\alpha(x,t_2,u,p)| \leq L|t_1 - t_2|^\gamma(1 + |u| + |p|).$$

Then for every weak solution u to (3.3) u_t *belongs to* $L_{2,loc}(Q)$.

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